ON THE LATIN CUBES OF THE SECOND ORDER AND THE FOURTH REPLICATION OF THE THREE-DIMENSIONAL OR CUBIC LATTICE DESIGNS

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1. INTRODUCTION AND SUMMARY

As is well known in the case of the simple or two-dimensional lattice designs, besides the two replications provided by taking the blocks along the rows and the columns of the scheme, additional replications may be generated by the superimposition of a system of orthogonal Latin squares. The object of the present paper is to extend this basic idea to designs in three dimensions, and to generate additional replications of the 3-dimensional or cubic lattice designs by superimposing Latin cubes of the second order as defined earlier by Kishen (1942, 1949) on the original lattice scheme.

Using the properties of Galois fields and finite geometries, Kishen (1949) gave a general method for constructing the s-sided, m-fold or $s \times s \times s \ldots$ (to m factors) Latin hypercubes of the r-th order, where $s = p^n$, p being a prime positive integer and n any positive integer. The present paper gives a new general method for constructing the 3-fold or $k \times k \times k$ Latin cubes of the second order of *any* side k whether prime or non-prime. A simple extension of this method leads to the construction of the 4-fold or, in general, the m-fold Latin hypercubes of the second order.

The analysis for k^3 varieties or treatments in k^2 blocks of k plots each, *i.e.*, for the (k^3, k^2) design, also known as a three-dimensional or cubic lattice, has been given for three replicates by Yates (1939); here k may be any of the integers 2, 3, 4, 5, 6, etc. In addition, he indicated the appropriate method of analysis for multiples of 3 replicates. Federer (1949), using the theory for prime power designs developed earlier by Kempthorne and Federer (1948), illustrated with a numerical example the analysis for p^3 varieties, where p is a prime number, in incomplete blocks of p varieties for more than 3 replicates. The computational procedures given by him use the pseudo- or quasi-factorial approach and are applicable for $p = 2, 3, 5, 7, 11$, etc., and for 4, 5, etc., replicates.

The present paper proves certain general properties of the Latin cubes of the second order in relation to partially balanced incomplete block designs. Following the general method of analysis for such designs as given by Bose and Nair (1939) and Rao (1947), the paper also develops a general solution for the cubic lattice designs with k^3 treatments and four replications arranged in blocks of k plots, where k is any integer and the fourth replication is generated by using suitable Latin cubes of the second order. The results are in perfect agreement with the theory as presented by Kempthorne (1952) in case k happens to be a prime number or its power. Finally, since orthogonal Latin cubes of the second order cannot be defined in three dimensions, the case of five or more replications is not amenable to this treatment. In the end, the results have been illustrated by re-working out Federer's numerical example with the methods presented in this paper.

2. Definitions and Notation

For the sake of completeness, the definition of an s-sided, m-fold Latin hypercube of the r-th order as given by Kishen (1949) is being reproduced here.

Definition.—An s-sided, m-fold or $s \times s \times s \ldots$ (to m factors) Latin hypercube of the r-th order may be defined as an m-fold hypercube arrangement of s^r letters, each repeated s^{m-r} times, such that each letter occurs exactly s^{m-r-1} times in each of its m sets of s, $(m-1)$-flats parallel to the m co-ordinate $(m-1)$-flats.

With $m = 3$ and $r = 2$ we arrive at the definition of a Latin cube of the second order as a cube arrangement of s^2 letters, each repeated s times, such that each letter occurs exactly once in each of its 3 sets of s planes parallel to the 3 co-ordinate planes.

For the partially balanced incomplete block, *i.e.*, P.B.I.B. designs with v varieties arranged in b blocks of k plots, each variety being replicated r times, we shall use the usual notation as given by Bose and Nair (1939) and Nair and Rao (1942). For a design with t associate classes, the first system of parameters will be denoted by

$$v, b, r, k;\ n_1, n_2, \ldots, n_t \text{ and } \lambda_1, \lambda_2, \ldots \lambda_t;$$

and the second system of parameters by the matrices

$$M \equiv (p^{\gamma}{}_{\alpha\beta})$$

where α, β, γ range from $1, 2, \ldots, t$.

For the analysis of these designs we shall use the method of Rao (1947) given later on in detail.

3. General Method of Construction of Latin Cubes of the Second Order of any Side k

We shall illustrate the general method with the help of an example for the $3 \times 3 \times 3$ Latin cubes of the second order. We require a cube arrangement of 3^2 letters, each repeated three times, so that each letter occurs exactly once in each of the 3 sets of three planes parallel to a co-ordinate plane.

Let the 3^2 letters be denoted by: 0, 1, 2, 3, 4, 5, 6, 7, 8. Divide them into 3 arbitrary groups containing 3 letters each. Let these groups be denoted by g_1, g_2, g_3, *e.g.*,

$$g_1: (0, 3, 6), \qquad g_2: (1, 4, 7), \qquad g_3: (2, 5, 8). \tag{1}$$

Arrange the three groups g_1, g_2, g_3 in the form of a Latin square, say:

$$\begin{matrix} g_1 & g_2 & g_3 \\ g_2 & g_3 & g_1 \\ g_3 & g_1 & g_2 \end{matrix} \tag{2}$$

Replacing g_i in (2) by the appropriate group of letters from (1), we arrive at the following scheme:

$$S_{123} \equiv \begin{Bmatrix} (0.3.6) & (1.4.7) & (2.5.8) \\ (1.4.7) & (2.5.8) & (0.3.6) \\ (2.5.8) & (0.3.6) & (1.4.7) \end{Bmatrix}. \tag{3}$$

The scheme S_{123} in (3) may be called the *generating scheme* of the $3 \times 3 \times 3$ Latin cubes of the second order. The different horizontal sections or *layers* of the cube are derived from S_{123} in succession. To obtain the first layer L_1, its first row is written down by selecting an arbitrary number from each of the three groups in the first row of the scheme S_{123}, *e.g.*,

Row 1 of L_1: 0 4 8 say.

Next, delete the three numbers 0, 4, 8 selected above, from all the groups in the rest of the scheme, wherever they occur, arriving at the following arrangement for constructing L_1:

$$\begin{Bmatrix} 0 & 4 & 8 \\ (1.7) & (2.5) & (3.6) \\ (2.5) & (3.6) & (1.7) \end{Bmatrix}. \tag{4}$$

The second row of L_1 is now obtained by selecting an arbitrary number from each group in the second row of scheme (4), *e.g.*,

Row 2 of L_1: 1 5 6 say.

Deleting 1, 5, 6 from each group in the last row of (4), the third row of L_1 is determined completely as:

Row 3 of L_1: 2 3 7.

Hence the first layer L_1 of the Latin cube becomes:

$$L_1: \begin{Bmatrix} 0 & 4 & 8 \\ 1 & 5 & 6 \\ 2 & 3 & 7 \end{Bmatrix}. \tag{5}$$

Next, delete each letter of L_1 in (5) from the group in the corresponding position of S_{123} in (3). This gives the following *residual* scheme for generating the second and third layers of the Latin cube:

$$S_{23} \equiv \begin{Bmatrix} (3.6) & (1.7) & (2.5) \\ (4.7) & (2.8) & (0.3) \\ (5.8) & (0.6) & (1.4) \end{Bmatrix}. \tag{6}$$

To generate L_2, the second layer of the Latin cube, the procedure outlined above for generating L_1 is repeated exactly. The first row of L_2 is chosen arbitrarily from the first row of the scheme S_{23}, *e.g.*,

Row 1 of L_2: 3 7 2 say.

Deleting 3, 7, 2 from the remainder of the scheme leads to

$$\begin{Bmatrix} 3 & 7 & 2 \\ 4 & 8 & 0 \\ (5.8) & (0.6) & (1.4) \end{Bmatrix}. \tag{7}$$

Since the second row of L_2 is also now fixed in (7) as 4 8 0, the last row of L_2 is clearly formed from the remaining numbers as 5 6 1. Hence the second layer L_2 of the Latin cube is found to be:

$$L_2: \left\{\begin{matrix} 3 & 7 & 2 \\ 4 & 8 & 0 \\ 5 & 6 & 1 \end{matrix}\right\}. \qquad (8)$$

Deleting each letter of L_2 in (8) from the corresponding cells of S_{23} in (6), we are left with the last layer L_3 of the Latin cube as follows:

$$S_3 \text{ or } L_3: \left\{\begin{matrix} 6 & 1 & 5 \\ 7 & 2 & 3 \\ 8 & 0 & 4 \end{matrix}\right\}. \qquad (9)$$

From (5), (8) and (9) the complete $3\times3\times3$ Latin cube of the second order may conveniently be represented diagrammatically by writing down the three layers L_1, L_2, L_3 side by side as follows:

$$\begin{matrix} & L_1 & & & L_2 & & & L_3 & \\ \end{matrix}$$
$$\left\{\begin{matrix} 0 & 4 & 8 & & 3 & 7 & 2 & & 6 & 1 & 5 \\ 1 & 5 & 6 & & 4 & 8 & 0 & & 7 & 2 & 3 \\ 2 & 3 & 7 & & 5 & 6 & 1 & & 8 & 0 & 4 \end{matrix}\right\}. \qquad (10)$$

It will be noticed that the Latin cube in (10) is identical with that given by Kishen (1949) in Table II of his paper. It is also evident that due to the arbitrary elements in the steps given above, (10) gives only one of the possible Latin cubes of the second order which can be generated from the scheme S_{123} in (3) which is itself arbitrary. As an example, we give below another Latin cube of the second order generated from the same scheme:

$$\begin{matrix} & L_1 & & & L_2 & & & L_3 & \\ \end{matrix}$$
$$\left\{\begin{matrix} 0 & 4 & 5 & & 3 & 1 & 8 & & 6 & 7 & 2 \\ 7 & 8 & 3 & & 4 & 2 & 6 & & 1 & 5 & 0 \\ 2 & 6 & 1 & & 5 & 0 & 7 & & 8 & 3 & 4 \end{matrix}\right\}. \qquad (11)$$

The method outlined above is completely general and can be used to generate $k\times k\times k$ Latin cubes of the second order of any side k where k is a prime or non-prime. The arrangement consists of k^2 letters each repeated k times, so that each letter occurs exactly once in each of the 3 sets of k planes parallel to a co-ordinate plane. The k^2 letters are first divided into k arbitrary groups

$$g_1, g_2, \ldots, g_k$$

each containing k letters. The groups 'g_i' are then arranged in the form of a $k \times k$ Latin square. This gives a generating scheme $S_{1, 2, \ldots k}$ for the $k \times k \times k$ Latin cubes of the second order. The first layer L_1 is generated by selecting an arbitrary first row from the first row of $S_{1, 2, \ldots k}$, deleting its elements from all the remaining groups, then forming an arbitrary second row from the second row of the resultant scheme, and so on exactly as outlined above. The elements of L_1 are next deleted from $S_{1, 2, \ldots k}$, to obtain the residual scheme $S_{2, 3, \ldots k}$, and the layers L_2, L_3, ... are successively generated in a similar manner until finally we reach L_{k-1} when L_k is automatically determined. Further examples of Latin cubes of the second order generated in this manner will be given in the subsequent sections.

It should also be pointed out that a simple repetition of the scheme (10) three times, one below the other, leads to the $3 \times 3 \times 3 \times 3$ or 4-fold Latin hypercube of the second order as given by Kishen (1949) in Table IX of his paper. An alternative is to arrange the layers L_1, L_2, L_3 in the form of a 3×3 Latin square in order to generate the 4-fold hypercube. The devices clearly apply to Latin cubes of the second order of any side k. The extension to higher dimensions may simply be derived by repetition of the schemes along the respective directions of the co-ordinate axes.

4. Latin Cubes of the Second Order in Relation to the Systems of Confounding in the (s^3, s^2) Design, where $s = p^n$

We may designate the s^3 varieties in the design by the symbols $(x_1\, x_2\, x_3)$ where the quantities x_1, x_2, x_3 can each take any one of the values $0, 1, 2, \ldots, s-1$, corresponding to the elements $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = x$, ..., $\alpha_{s-1} = x^{s-2}$ of the Galois field $GF(s)$ of $s = p^n$ elements where p is a prime number and n any positive integer.

Now, let the numbers $0, 1, 2, \ldots, (s-1)$ be written in order by proceeding systematically along each of the three co-ordinate axes OX_1, OX_2, OX_3. Then the sets $(x_1\, x_2\, x_3)$ represent the co-ordinates of the s^3 points in a three-dimensional lattice. The point $(x_1\, x_2\, x_3)$ may be spoken of as the *cell* $(x_1\, x_2\, x_3)$ of the three-dimensional $(s \times s \times s)$-

cube arrangement, and the s different layers of the cube are given by the equations $x_3 = 0, 1, 2, \ldots, (s-1)$.

Finally we may take x_1, x_2, x_3 as representing the s levels of three quasi-factors a, b, c respectively. It is then evident that the s^3 varieties, lattice points or cells of the cube are in (1, 1) correspondence with the s^3-treatment combinations of a quasi-factorial system. Hence, in accordance with the general theory of confounding, the sets of $(s-1)$ degrees of freedom belonging to the main effects and interactions of the quasi-factors are obtained by the contrasts of the s sets of s^2 treatment combinations satisfying equations of the following type in $GF(s)$ as given in Table I.

TABLE I

Nature of effects and corresponding equations in confounded designs involving three factors

Effect or Interaction	Equations
A	.. $\alpha_{x_1} = \alpha_t \; (t = 0, 1, 2, \ldots, s-1)$
$AB^i \; (i = 1, 2, \ldots, s-1)$	.. $\alpha_{x_1} + \alpha_i \alpha_{x_2} = \alpha_t \; (t = 0, 1, 2, \ldots, s-1)$
$AB^iC^j \; (i, j = 1, 2, \ldots, s-1)$	.. $\alpha_{x_1} + \alpha_i \alpha_{x_2} + \alpha_j \alpha_{x_3} = \alpha_t \; (t = 0, 1, 2, \ldots, s-1)$

The notation simplifies when $s = p$, a prime number; for, in this case x_1, x_2, x_3 are the actual elements $0, 1, 2, \ldots, p-1$ of the Galois field $GF(p)$, and the equations in Table I may be written in the form:

$$\left.\begin{aligned} x_1 &= t \\ x_1 + ix_2 &= t \\ x_1 + ix_2 + jx_3 &= t, \; (t = 0, 1, 2, \ldots, p-1) \end{aligned}\right\} \quad (12)$$

remembering that all operations have now to be performed mod. (p). We will also use the two following well-known results:

(a) The (s^3, s^2) design has only two independent generators, *i.e.*, we can confound any two arbitrary sets of $(s-1)$ d.f., given by any two sets of equations chosen from Table I. If these confounded effects are denoted by X and Y, then the effects:

$$XY,\ XY^2, \ldots, XY^{s-1} \tag{13}$$

representing their generalised interaction are also confounded.

(*b*) The total number of possible systems of confounding for an (s^3, s^2) experiment in blocks of size s is equal to:

$$\frac{(s^3-1)\ (s^3-s)}{(s^2-1)\ (s^2-s)} = s^2 + s + 1. \tag{14}$$

It is now of interest to see the relationship of the different possible systems of confounding to the Latin cubes of the second order. As a simple example, we will first consider the case of the 3^3 lattice design in blocks of 3 plots. It is well known that Yates' 3 replications of the cubic lattice—for any side—are simply generated by taking the blocks along the rows, columns and verticals of the lattice scheme, and that these correspond to the following systems of confounding:

Replicate	Effects confounded				
I	..	A,	B,	AB,	AB^2
II	..	A,	C,	AC,	AC^2
III	..	B,	C,	BC,	BC^2

(15)

These replicates have been designated as Z, Y and X respectively by Yates. Suppose that in the fourth replication, as in Federer's (1949) example, we wish to adopt the following confounding:

$$\text{Replicate IV: } AB^2,\ BC,\ AC,\ ABC^2. \tag{16}$$

Now any two of the effects given in (16) may be taken as the generators of the design. We thus have to solve the following two sets of three equations in $GF(3)$ to obtain the blocks of the design:

$$\left.\begin{aligned} x_1 + 2x_2 &= 0, 1, 2 \\ x_2 + x_3 &= 0, 1, 2 \end{aligned}\right\} \tag{17}$$

corresponding to AB^2 and BC as generators. The solutions of (17) corresponding to the nine combinations on the right-hand side readily give the following nine blocks of 3 varieties each for the fourth replication of the 3^3 lattice:

Block No.	Varieties
(1)	(000,112,221)
(2)	(010,122,201)
(3)	(020,102,211)
(4)	(100,212,021)
(5)	(110,222,001)
(6)	(120,202,011)
(7)	(200,012,121)
(8)	(210,022,101)
(9)	(220,002,111)

(18)

The 3^3 varieties corresponding to the 3^3 cells of the lattice may now be written in the following scheme, where as before, the layers corresponding to $x_3 = 0$, 1, and 2 are given side by side for convenience:

$$\begin{array}{ccccccccc} & x_3 = 0 & & & x_3 = 1 & & & x_3 = 2 & \\ \left\{\begin{array}{ccc} 020 & 120 & 220 \\ 010 & 110 & 210 \\ 000 & 100 & 200 \end{array}\right. & & & \begin{array}{ccc} 021 & 121 & 221 \\ 011 & 111 & 211 \\ 001 & 101 & 201 \end{array} & & & \left.\begin{array}{ccc} 022 & 122 & 222 \\ 012 & 112 & 212 \\ 002 & 102 & 202 \end{array}\right\} & & \end{array} . \quad (19)$$

The blocks may be numbered serially in any manner from 1, 2, . . ., 9. If now in scheme (19) we write the number 'i' in the three positions corresponding to the three varieties occurring in the i-th block ($i = 1$, 2, . . ., 9), we generate the following scheme:

$$\begin{array}{ccc} L_1 & L_2 & L_3 \\ \left\{\begin{array}{ccc} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{array}\right. & \begin{array}{ccc} 4 & 7 & 1 \\ 6 & 9 & 3 \\ 5 & 8 & 2 \end{array} & \left.\begin{array}{ccc} 8 & 2 & 5 \\ 7 & 1 & 4 \\ 9 & 3 & 6 \end{array}\right\} \end{array} . \quad (20)$$

It is evident that the scheme (20) thus obtained is a $3 \times 3 \times 3$ Latin cube of the second order.

The procedure described above for 3^3 varieties is quite general and may be followed without change for s^3 varieties. In the general case, we may select any two effects, each representing $(s - 1)$ d.f., out of the $(s + 1)$ effects confounded in the fourth replication, as the generators of the design. The two sets of s equations in α_{x_1}, α_{x_2}, α_{x_3}

in $GF(s)$ for these two effects are written down with their right-hand sides as $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$. The s solutions to these pairs of equations for each of the s^2 combinations of the right-hand sides generate the s^2 blocks of the design, which may be numbered serially as $1, 2, \ldots, s^2$ in any order. In the s cells $(x_1\, x_2\, x_3)$ of an $s \times s \times s$ cubic lattice corresponding to the s varieties occurring in the i-th block $(i = 1, 2, \ldots, s^2)$, the number 'i' is written, leading to the generation of an $s \times s \times s$ Latin cube of the second order. The process, however, breaks down if a main effect is included among the confounded effects.

It will be readily recognised that the above procedure is identical with the general result proved by Kishen (1949) for generating the m-fold Latin hypercubes of the second order, for the particular case $m = 3$. In general, starting with two equations in $\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_m}$ and right-hand sides α_{t_1} and α_{t_2} in $GF(s)$, a number corresponding to the pair (t_1, t_2) is written in each of the s^{m-2} cells $(x_1, x_2, \ldots, x_m)$ of the hypercube whose co-ordinates satisfy this pair of equations. The numbers $0, 1, 2, \ldots, (s^2 - 1)$ are attached to the pairs (t_1, t_2) in any convenient manner. It is, however, essential that the following condition must be satisfied in order that the process may generate Latin cubes (or hypercubes) of the second order. The coefficients of *at least two* of the α_{x_i} on the left-hand side of each of the two equations must be non-zero (Kishen, 1949). Using this result it will be seen that, in order to be able to generate Latin cubes of the second order by the method described above, no *main effect* should be included among the effects confounded in the fourth replication of the designs under consideration.

We shall now analyse in some more detail the $(s^2 + s + 1)$ possibilities of confounding in the s^3 lattice designs in blocks of s plots (*cf.*, Bose and Kishen, 1940; Bose, 1947; and Kempthorne, 1952). The $(s^2 + s + 1)$ cases may be easily classified into the three types as given in Table II.

The subdivision of the possible types of confounding given in Table II corresponds in essence to that given by Bose and Kishen (1940) in Table IV on the basis of the geometrical theory of confounding. It will now be seen that the 3 possibilities of confounding of Type I correspond to the 3 basic replications of the cubic lattice designs where the blocks are formed by taking all varieties lying along the rows, columns and verticals of a 3-dimensional cube. The $3(s - 1)$ possibilities of confounding of Type II involve the confounding of a main effect and hence, in virtue of the result quoted above, are not suitable for a fourth replication, as these will not lead to Latin cubes of the second order. The $(s-1)^2$ possibilities of confounding of Type III are all that we are left with to generate the fourth replication of the cubic lattice designs leading to the construction of $s \times s \times s$ Latin cubes of the second order.

TABLE II

Nature of confounding in the (s^3, s^2) *designs*

Type	Effects confounded	Number of cases
I	$A, B, AB, AB^2, \ldots, AB^{s-1}$	
	$A, C, AC, AC^2 \ldots, AC^{s-1}$	3
	$B, C, BC, BC^2, \ldots, BC^{s-1}$	
II	A, BC^i, with products giving the generalised interactions $(i = 1, 2, \ldots, s-1)$	$(s-1)$
	B, AC^i, with products $(i = 1, 2, \ldots, s-1)$	$(s-1)$
	C, AB^i, with products $(i = 1, 2, \ldots, s-1)$	$(s-1)$
III	AB^i, BC^j, with products, $(i, j = 1, 2, \ldots, s-1)$	$(s-1)^2$
	TOTAL ..	(s^2+s+1)

We shall conclude the section with the following remarks which may be easily verified:—

(*i*) The confounding in Type III in Table II may alternatively be represented by the symbols:

$$AB^i, \ AC^k, \text{ with products. } \ (i, \ k = 1, \ 2, \ldots, \ s-1) \qquad (21)$$

(*ii*) Among the $(s+1)$ effects confounded with any particular system of Type III, 3 effects belong one each to the first order interactions AB, BC, AC and $(s-2)$ effects belong to the second order interaction ABC.

We shall now prove certain further general properties of the Latin cubes of the second order generated by these confounded designs in the next section.

5. Some General Properties of Latin Cubes of the Second Order Generated by the (s^3, s^2) Lattice Designs

We have seen in the previous section that out of the possible $(s^2 + s + 1)$ systems of confounding, only $(s - 1)^2$ systems belonging to Type III lead to the construction of Latin cubes of the second order. These systems, taking AB^i and BC^j as generators, are characterised by the following two sets of equations:

$$\left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_{x_2} &= \alpha_{t_1}\,(t_1 = 0, 1, 2, \ldots, s-1) \\ \alpha_{x_2} + \alpha_j\alpha_{x_3} &= \alpha_{t_2}\,(t_2 = 0, 1, 2, \ldots, s-1)\ i, j \neq 0 \end{aligned}\right\}. \qquad (22)$$

For any one system of confounding, the subscripts i and j in (22) are fixed and non-zero, and if they are allowed to range from 1, 2, ..., $s - 1$, all the $(s - 1)^2$ systems of confounding belonging to Type III are generated. We shall now prove four important properties satisfied by Latin cubes of the second order generated by the equations (22).

Theorem 1.—*In a given Latin cube of the second order of side* $s = p^n$, p *being a prime number, the row* (*or column*) *contents in each layer of the cube* (i.e., *in all planes* x_3 = *constant*) *are identical. The sets constituting the rows* (*or columns*) *as well as the numbers within a set are, however, arranged in a different order in the different layers.*

We shall prove the result for the row contents in any two layers of the Latin cube. The result for columns may be established in a similar manner by considering the system of equations appropriate to the confounding of effects in the form (21).

Consider any two layers L_u and L_v of the Latin cube given by $x_3 = u$ and $x_3 = v$, where u, v are any two numbers out of 0, 1, 2, ..., $(s - 1)$. The s rows in these layers are generated by successively putting $x_2 = 0, 1, 2, \ldots, s - 1$. Hence we get the following s pairs of equations for determining the row contents in the two layers:

$$\left.\begin{array}{ccc} \text{Row No.} & \text{Layer } L_u & \text{Layer } L_v \\ 0 & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_0 &= \alpha_{t_1} \\ \alpha_0 + \alpha_j\alpha_u &= \alpha_{t_2} \end{aligned}\right\} & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_0 &= \alpha_{t_1} \\ \alpha_0 + \alpha_j\alpha_v &= \alpha_{t_2} \end{aligned}\right\} \\ 1 & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_1 &= \alpha_{t_1} \\ \alpha_1 + \alpha_j\alpha_u &= \alpha_{t_2} \end{aligned}\right\} & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_1 &= \alpha_{t_1} \\ \alpha_1 + \alpha_j\alpha_v &= \alpha_{t_2} \end{aligned}\right\} \\ \cdots & \cdots & \cdots \\ (s-1) & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_{s-1} &= \alpha_{t_1} \\ \alpha_{s-1} + \alpha_j\alpha_u &= \alpha_{t_2} \end{aligned}\right\} & \left.\begin{aligned} \alpha_{x_1} + \alpha_i\alpha_{s-1} &= \alpha_{t_1} \\ \alpha_{s-1} + \alpha_j\alpha_v &= \alpha_{t_2} \end{aligned}\right\} \end{array}\right\}. \qquad (23)$$

In the above equations the numbers i, j, u, v are all fixed. By giving x_1 the s values 0, 1, 2, ..., $s-1$ in any pair of equations, the pairs (t_1, t_2) are determined and hence the s numbers in this row become known. Now, let

$$(\alpha_j \alpha_u) - (\alpha_j \alpha_v) = \alpha_r, \quad \text{say.}$$

We then have

$$\alpha_\lambda + \alpha_j \alpha_u = \alpha_j \alpha_v + \alpha_r + \alpha_\lambda \quad (\lambda = 0, 1, 2, \ldots, s-1) \tag{24}$$

Since α_r added to all the s elements of the Galois field generates the same s elements in some different order, the relation (24) becomes:

$$\alpha_\lambda + (\alpha_j \alpha_u) = (\alpha_j \alpha_v) + \alpha_\mu, \; (\lambda \neq \mu). \tag{25}$$

It then follows from (23) and (25) that the equation determining the number 't_2' in any row of L_u is identical with the equation determining 't_2' in some different row of L_v. In this manner each row in L_u corresponds to some other unique row of L_v and *vice-versa* through a common number 't_2'. Now consider any two such rows corresponding to different values of x_2, say $x_2 = p$ and $x_2 = q$ where $p \neq q$, one from L_u and one from L_v, which give the same value for 't_2'. The equations giving the values of 't_1' for these two rows may then be written as follows:

$$p\text{-th row of } L_u: \; \alpha_{x_1} + \alpha_i \alpha_p = \alpha_{t_1} \tag{26}$$

$$q\text{-th row of } L_v: \; \alpha_{x_1} + \alpha_i \alpha_q = \alpha_{t_1}. \tag{27}$$

It is now evident from the properties of the Galois field that on letting $x_1 = 0, 1, 2, \ldots, s-1$, the number '$t_1$' in each of the equations (26) and (27) ranges through all the values 0, 1, 2, ..., $s-1$. Since the number 't_2' is fixed and common to both these rows it follows that the contents of the p-th row of L_u and the q-th row of L_v correspond to the same set of pairs (t_1, t_2) and hence are identical. Since $p \neq q$, it also follows from (26) and (27) that the same value of t_1 must correspond to different values of x_1. Hence a given number in the two rows must occur in a *different* column, *i.e.*, in a different position. This proves that the rows in any two layers of a Latin cube generated by (22) are identical, the corresponding rows occupying a different position in the layer, and the corresponding numbers in these rows occurring in a different order. Hence the theorem.

The following two corollaries may easily be proved in the same manner:

Corollary 1.—The row content is the same in all the $(s-1)^2$ Latin cubes of the second order generated by (22).

Corollary 2.—The column content is the same in all the $(s-1)$ Latin cubes of the second order, corresponding to a fixed value of i and generated by (22).

THEOREM 2.—*Among the* $(s-1)$ *Latin cubes of the second order generated by confounding the effects* AB^i, BC^j $(i = \text{fixed}, j = 1, 2, \ldots, s-1)$, *the first layer* $(x_3 = 0)$ *is identical. The other layers are also identical but they are only arranged in a different order.*

Consider any two Latin cubes of the second order corresponding to the confounding of the effects:

(*a*) AB^i, BC^{j_1}, and generalised interactions,

(*b*) AB^i, BC^{j_2}, and generalised interactions,

where $j_1 \neq j_2 =$ one of the numbers $1, 2, \ldots, s-1$.

The equations to the different layers of the Latin cubes (*a*) and (*b*) may then be written as follows:

Layer No.	$x_3 = 0$	$x_3 = 1$	$\ldots$ $x_3 = s-1$
Latin cube (*a*) ..	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} = \alpha_{t_2} \end{matrix}\right\}$	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} + \alpha_{j_1}\alpha_1 = \alpha_{t_2} \end{matrix}\right\}$	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} + \alpha_{j_1}\alpha_{s-1} = \alpha_{t_2} \end{matrix}\right\}$
Latin cube (*b*) ..	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} = \alpha_{t_2} \end{matrix}\right\}$	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} + \alpha_{j_2}\alpha_1 = \alpha_{t_2} \end{matrix}\right\}$	$\left.\begin{matrix} \alpha_{x_1} + \alpha_i\alpha_{x_2} = \alpha_{t_1} \\ \alpha_{x_2} + \alpha_{j_2}\alpha_{s-1} = \alpha_{t_2} \end{matrix}\right\}$.

It will be noticed that the equations for the first layer L_0 are identical for the two Latin cubes. In the remaining layers $L_1, L_2, \ldots, L_{s-1}$ the 't_1' equation is identical throughout. For the 't_2' equation, it may be noted that in the two sets of products:

(*i*) $\alpha_{j_1}\alpha_1, \alpha_{j_1}\alpha_2, \ldots, \alpha_{j_1}\alpha_{s-1}$,

(*ii*) $\alpha_{j_2}\alpha_1, \alpha_{j_2}\alpha_2, \ldots, \alpha_{j_2}\alpha_{s-1}$,

each set gives all the non-zero elements of the Galois field but in a different order. Hence the 't_2' equation for any layer of the Latin cube (*a*) is identical with the 't_2' equation of some other layer of the

Latin cube (*b*). It follows that the pair of equations for any layer of Latin cube (*a*) is identical with the pair of equations of some other layer of Latin cube (*b*). Thus a selected pair of values of (x_1, x_2) leads to the same values of (t_1, t_2), *i.e.*, the same number out of 0, 1, 2, ..., $s^2 - 1$ occurs in the same position in these two layers belonging to the respective Latin cubes of the second order. Hence the theorem. It may be remarked that a rearrangement of the layers corresponds to a change in the name of the levels of the factor '*c*'.

THEOREM 3.—*Among the* (s − 1) *Latin cubes of the second order generated by confounding the effects* AB^i, AC^k (i = fixed, k = 1, 2, ..., s − 1), *one, corresponding to the case* k = i, *is invariant under the transformation* (x_2x_3). *The remaining* (s − 2) *Latin cubes of this set, on application of the transformation* (x_2x_2), *become identical with a Latin cube belonging to each one of the* (s − 2) *sets corresponding to the confounding of* AB^j, AC^k ($j \neq i = 1, 2, \ldots, s-1$; $k = 1, 2, \ldots, s-1$), *so that the same sets of effects are confounded in them.*

For convenience in writing let us fix $i = 1$, so that $j = 2, 3, \ldots, s - 1$. Equations of sets of $(s - 1)$ Latin cubes corresponding to systems of confounding for these values of i and j may then be written as follows:

Set $i = 1$	Set $j = 2$	...	Set $j = s - 1$
$\left.\begin{aligned} a_{x_1} + a_1 a_{x_2} &= a_{t_1} \\ a_{x_1} + a_k a_{x_3} &= a_{t_2} \end{aligned}\right\}$	$\left.\begin{aligned} a_{x_1} + a_2 a_{x_2} &= a_{t_1} \\ a_{x_1} + a_k a_{x_3} &= a_{t_2} \end{aligned}\right\}$		$\left.\begin{aligned} a_{x_1} + a_{s-1} a_{x_2} &= a_{t_1} \\ a_{x_1} + a_k a_{x_3} &= a_{t_2} \end{aligned}\right\}$
$(k = 1, 2, \ldots, s - 1)$	$(k = 1, 2, \ldots, s - 1)$		$(k = 1, 2, \ldots, s - 1)$
(C_{1k})	(C_{2k})		$(C_{s-1.k})$

The Latin cubes belonging to these sets may be systematically numbered as $C_{1k}, C_{2k}, \ldots, C_{(s-1)\cdot k}$ where k ranges from 1, 2, ... $s - 1$. It is now evident from the equations written above that the first Latin cube C_{11} of the first set is invariant under the transformation (x_2x_3). Applying the transformation (x_2x_3) to the remaining $(s - 2)$ Latin cubes of the first set, the following equivalences are immediately apparent:

$$C_{12} = C_{21},\ C_{13} = C_{31},\ \ldots,\ C_{1.(s-1)} = C_{(s-1).1} \tag{28}$$

In other words, the 2nd, 3rd, ..., $(s - 1)$-th Latin cubes of the *first set* become identical with the *first* Latin cube (corresponding to $k = 1$) of the 2nd set, 3rd set, ..., $(s - 1)$-th set respectively. Hence the theorem.

The interchange of the letters x_2 and x_3 is equivalent to the interchange of the factors b and c. Hence, in the Latin cubes shown to be equivalent above, the *d.f.*, confounded in one are identical with those confounded in the other on application of the transformation (BC).

THEOREM 4.—*The* (s − 1) *Latin cubes of the second order corresponding to the confounding of* AB^i, BC^j (j = fixed, $i = 1, 2, \ldots, s-1$) *have the same first row plane*, i.e., *the first rows in all the corresponding layers are identical.*

The result is immediately apparent on putting $x_2 = 0$ in the equations (22), when these become

$$\left.\begin{array}{l} \alpha_{x_1} = \alpha_{t_1} \\ \alpha_j \alpha_{x_3} = \alpha_{t_2} \end{array} \quad (t_1, t_2 = 0, 1, 2, \ldots, s-1)\right\} . \qquad (29)$$

The equations (29) do not involve α_i and α_j is fixed. Hence this row plane is the same in all the Latin cubes corresponding to the different values of $i = 1, 2, \ldots, s-1$.

Illustration.—As a simple illustration of the properties proved in the four theorems above consider the 3^3 lattice in blocks of 3 plots where there are only $(3-1)^2 = 4$ systems of confounding which lead to Latin cubes of the second order. These are as follows:

(1) AB, BC^2, AC, AB^2C^2

(2) AB, BC, AC^2, AB^2C

(3) AB^2, BC, AC, ABC^2

(4) AB^2, BC^2, AC^2, ABC

Adopting the identification:

Pair (t_1, t_2) ..	(00)	(01)	(02)	(10)	(11)	(12)	(20)	(21)	(22)
No. attached ..	1	2	3	4	5	6	7	8	9,

the four Latin cubes of the second order, say C_{11}, C_{12}, C_{21}, C_{22} respectively, are obtained as follows:

		L_1			L_2			L_3		
C_{11}	..	9	3	6	8	2	5	7	1	4
		5	8	2	4	7	1	6	9	3
		1	4	7	3	6	9	2	5	8
C_{12}	..	9	3	6	7	1	4	8	2	5
		5	8	2	6	9	3	4	7	1
		1	4	7	2	5	8	3	6	9
C_{21}	..	6	9	3	4	7	1	5	8	2
		8	2	5	9	3	6	7	1	4
		1	4	7	2	5	8	3	6	9
C_{22}	..	6	9	3	5	8	2	4	7	1
		8	2	5	7	1	4	9	3	6
		1	4	7	3	6	9	2	5	8

It will be noted that the row contents in any layer of any Latin cube are constant. The column contents are also constant in the sets C_{11}, C_{12} and C_{21}, C_{22} corresponding to the fixed values of $i = 1$ and $i = 2$. The layer L_1 in the set C_{11}, C_{12} is identical; also L_2 and L_3 are only placed in a different order. The same holds for the set C_{21}, C_{22}. Further, the Latin cubes C_{11} and C_{22}—*i.e.*, the d.f., confounded in them—are invariant under the transformation (BC), while the d.f. confounded in C_{12} and C_{21} are seen to be identical on making the transformation (BC). Lastly, the first (lowermost) rows in the Latin cubes C_{12}, C_{21} ($j = 1$) and C_{11}, C_{22} ($j = 2$) are seen to be the same.

6. Latin Cubes of the Second Order in Relation to P.B.I.B. Designs

We shall begin by stating the different combinatorial relationships satisfied by the parameters of the partially balanced incomplete block designs with 't' associate classes. In the usual notation, we have,

$$\left.\begin{array}{l} n_1 + n_2 + n_3 + \ldots + n_t = v - 1 \\ n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + \ldots + n_t\lambda_t = r(k-1) \\ p^{\alpha}{}_{\beta\gamma} = p^{\alpha}{}_{\gamma\beta} \\ n_\alpha p^{\alpha}{}_{\beta\gamma} = n_\beta p^{\beta}{}_{\alpha\gamma} = n_\gamma p^{\gamma}{}_{\alpha\beta} \\ \sum_{\gamma=1}^{t} p^{\alpha}{}_{\beta\gamma} = n_\beta - 1 \text{ or } n_\beta \text{ according as } \alpha = \beta \text{ or } \alpha \neq \beta \end{array}\right\} . \quad (30)$$

Consider now the (k^3, k^2) lattice designs in blocks of k plots and with four replications. In order to determine their relationship with p.b.i.b. designs we shall consider the cases (A) $k = s =$ a prime number or its power, and (B) $k =$ non-prime, separately.

Case A.—k = s *is a prime number or its power*

We shall assume that in all cases the blocks of the fourth replication are generated by superimposing on the s^3 lattice scheme giving the variety numbers an $s \times s \times s$ Latin cube of the second order constructed with the help of the usual theory of confounding for the (s^3, s^2) designs. We have seen is Section 4, that there are only $(s-1)^2$ such possibilities of confounding of the type AB^i, BC^j $(i, j = 1, 2, \ldots, s-1)$ which lead to the construction of Latin cubes of the second order of side s. The $(s-1)^2$ Latin cubes of the second order thus generated may now be divided into $(s-1)$ groups $G_1, G_2, \ldots, G_{s-1}$, of $(s-1)$ Latin cubes each, corresponding to the fixed values of $i = 1$, $i = 2, \ldots, i = s-1$ and $j = 1, 2, \ldots, s-1$, respectively. Now from Theorems 2 and 3 proved in the last section we see that:

(*i*) The $(s-1)$ Latin cubes of the second order in any group G_i for a fixed value of i correspond only to a rearrangement of the layer planes $x_3 =$ constant, *i.e.*, only to a change in the names of the levels of the quasi-factor c.

(*ii*) The groups $G_1, G_2, \ldots, G_{s-1}$ are interrelated by the transformation (x_2, x_3), *i.e.*, by a change in the name of the two quasi-factors b and c.

Hence we deduce that the $(s-1)^2$ possible Latin cubes of the second order, generated by the general theory of confounding, are all structurally identical and so it is sufficient to consider the combinatorial properties of only one of them.

We shall now consider the cases $s = 3, 4, 5, 7, \ldots;$ individually. It is then found that the 3^3 lattice with four replications—using any one of the $(3-1)^2 = 4$ possible Latin cubes of the second order for generating the blocks of the fourth replication—is a p.b.i.b. design with 3 associate classes. We assume that a Latin cube of the second order, *e.g.*, the one given in scheme (20) is superimposed on the lattice scheme (19) giving the variety numbers. Then the association scheme of any variety (x_1, x_2, x_3), occurring in the r-th row and c-th column of the l-th layer and corresponding to the number 'i' of the Latin cube is given by the following rules:

(*a*) *First associates*:

In the l-th layer —Varieties in the same row and column as (x_1, x_2, x_3).

In the other layers—Varieties at the position (r, c) in the layers, *i.e.*, lying along the vertical through (x_1, x_2, x_3);

—Varieties in the positions occupied by the number 'i'.

(*b*) *Second associates*:

In the l-th layer —Varieties corresponding to the projections of 'i' occurring in the remaining layers on to the l-th layer.

In the other layers—Varieties corresponding to the projections of 'i' in any layer on to the r-th row and c-th column in this layer.

(*c*) *Third associates*:

In the l-th layer —All varieties remaining after the first and second associates are written down.

In the other layers—All varieties occurring in the two rows and two columns determined by the two first and two second associates in this layer, excluding the first and second associates;

—Varieties corresponding to the projection of 'i', from all layers excluding the l-th, on to this layer.

Thus, from the schemes (19) and (20), the three associate categories of the variety (000) for the design specified by confounding the effects given in (16) in the fourth replication are as follows:

$$\left.\begin{array}{ll} \text{1st associates}: & 100,\ 200,\ 010,\ 020,\ 001,\ 221,\ 002,\ 112 \\ \text{2nd associates}: & 110,\ 220,\ 021,\ 201,\ 012,\ 102 \\ \text{3rd associates}: & 120,\ 210,\ 011,\ 101,\ 111,\ 121,\ 211,\ 022, \\ & 122,\ 202,\ 212,\ 222 \end{array}\right\} . \tag{31}$$

The categories (31) are seen to be the same as those obtained by Federer (1949).

The varieties forming the first associates occur together in a block with (000). The other varieties do not occur together in a block with (000). The parameters of the $(3^3, 3^2)$ lattice with 4 replications are then found to be as follows:

$$\left.\begin{array}{l} v = 3^3 = 27, \quad b = 36, \quad r = 4, \quad k = 3 \\ n_1 = 8 \quad n_2 = 6 \quad n_3 = 12 \\ \lambda_1 = 1 \quad \lambda_2 = 0 \quad \lambda_3 = 0 \\ M_1 \equiv (p^1{}_{\alpha\beta}) = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 6 \end{pmatrix}, \\ M_2 \equiv (p^2{}_{\alpha\beta}) = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 4 \\ 4 & 4 & 4 \end{pmatrix}, \\ M_3 \equiv (p^3{}_{\alpha\beta}) = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix} \end{array}\right\} . \tag{32}$$

Consider next the 4^3 lattice with four replications, using any one of the $(4-1)^2 = 9$ possible Latin cubes of the second order for generating the blocks in the 4-th replication. This is found to be a p.b.i.b. design with four associate classes. Assume as before the Latin cube of the second order selected for generating the design to be superimposed on a $4 \times 4 \times 4$ lattice scheme giving the variety numbers. The first three associates of any variety (x_1, x_2, x_3) occurring in the l-th layer are then given by the rules (*a*), (*b*), (*c*) given above, and the fourth associates are given by the following rule:

(*d*) *Fourth associates*:

In the l-th layer —Nil

In the other layers—All varieties remaining in the layers after excluding the 1st, 2nd and 3rd associates in these layers.

As an example, consider the following $4 \times 4 \times 4$ Latin cube of the second order generated by confounding the effects:

$$AB,\ BC,\ AC,\ AB^2C^3,\ AB^3C^2 \tag{33}$$

in the fourth replication.

$$\begin{Bmatrix} \multicolumn{4}{c}{L_1} & \multicolumn{4}{c}{L_2} & \multicolumn{4}{c}{L_3} & \multicolumn{4}{c}{L_4} \\ 4 & 8 & 12 & 16 & 7 & 3 & 15 & 11 & 10 & 14 & 2 & 6 & 13 & 9 & 5 & 1 \\ 3 & 7 & 11 & 15 & 8 & 4 & 16 & 12 & 9 & 13 & 1 & 5 & 14 & 10 & 6 & 2 \\ 2 & 6 & 10 & 14 & 5 & 1 & 13 & 9 & 12 & 16 & 4 & 8 & 15 & 11 & 7 & 3 \\ 1 & 5 & 9 & 13 & 6 & 2 & 14 & 10 & 11 & 15 & 3 & 7 & 16 & 12 & 8 & 4 \end{Bmatrix}. \tag{34}$$

Superimposing the Latin cube (34) on a $4 \times 4 \times 4$ lattice similar to the $3 \times 3 \times 3$ lattice scheme (19) giving the variety numbers, we find on applying the rules (*a*), (*b*), (*c*), (*d*) given above that the different associates of the variety (000) are as follows:

1st associates: 010, 020, 030, 100, 200, 300, 001, 111, 002, 222, 003, 333.

2nd associates: 110, 220, 330, 011, 101, 022, 202, 033, 303.

3rd associates: 120, 130, 210, 230, 310, 320, 021, 031, 121, 131, 201, 211, 221, 301, 311, 331, 012, 032, 102, 112, 122, 212, 232, 302, 322, 332, 013, 023, 103, 113, 133, 203, 223, 233, 313, 323.

4th associates: 231, 321, 132, 312, 123, 213.

The parameters of the $(4^3, 4^2)$ lattice with four replications are then found to be as follows:

$$v = 4^3 = 64, \quad b = 64, \quad r = 4, \quad k = 4$$

$$n_1 = 12, \quad n_2 = 9, \quad n_3 = 36, \quad n_4 = 6$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0.$$

$$M_1 \equiv (p^1{}_{\alpha\beta}) = \begin{pmatrix} 2 & 3 & 6 & 0 \\ 3 & 0 & 6 & 0 \\ 6 & 6 & 18 & 6 \\ 0 & 0 & 6 & 0 \end{pmatrix}, \qquad M_2 \equiv (p^2{}_{\alpha\beta}) = \begin{pmatrix} 4 & 0 & 8 & 0 \\ 0 & 4 & 0 & 4 \\ 8 & 0 & 28 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$$

$$M_3 \equiv (p^3{}_{\alpha\beta}) = \begin{pmatrix} 2 & 2 & 6 & 2 \\ 2 & 0 & 7 & 0 \\ 6 & 7 & 18 & 4 \\ 2 & 0 & 4 & 0 \end{pmatrix}, \qquad M_4 \equiv (p^4{}_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 12 & 0 \\ 0 & 6 & 0 & 3 \\ 12 & 0 & 24 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix}.$$

Consider now the s^3 lattice designs with side $s > 4$, and using any one of the $(s - 1)^2$ possible Latin cubes of the second order for generating the blocks of the fourth replication. It is found that these designs also have four associate classes, but the parameters do not satisfy all the relationships (30) which must hold in a p.b.i.b. design. In particular, the relation

$$n_\alpha p^\alpha{}_{\beta\gamma} = n_\beta p^\beta{}_{\alpha\gamma} = n_\gamma p^\gamma{}_{\alpha\beta} \tag{35}$$

for parameters of the second kind is violated for the matrices M_2, M_3, M_4 except for their first rows (or columns). All other relations, including (35) are satisfied for the matrix M_1 and the first rows of the matrices M_2, M_3, M_4.

It will further be seen from the general solution for p.b.i.b. designs with four associate classes and $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, presented in the next section that it is only the elements of the *first rows* of the matrices M_γ $(\gamma = 1, 2, 3, 4)$ that enter into the normal equations. Hence it follows that this solution applies without change to the (s^3, s^2) lattices with $s > 4$. Although some of the p.b.i.b. properties do not hold for these designs the solutions are unaffected by virtue of all λ's excepting the first one being zero. We shall, therefore, give below the general values of the parameters for the $s \times s \times s$ cubic lattice designs with four replications. These are as follows:

$$\left.\begin{aligned} & v = s^3, \quad b = 4s^2, \quad r = 4, \quad k = s \\ & n_1 = 4(s-1), \quad n_2 = 3(s-1), \quad n_3 = 6(s-1)(s-2), \\ & n_4 = (s-1)(s-2)(s-3) \\ & \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0 \end{aligned}\right\}. \tag{36}$$

$$M_1 \equiv (p^1{}_{\alpha\beta})$$

$$= \begin{pmatrix} (s-2) & 3 & 3(s-2) & 0 \\ 3 & 0 & 3(s-2) & 0 \\ 3(s-2) & 3(s-2) & 3(s-1)(s-2) & 3(s-2)(s-3) \\ 0 & 0 & 3(s-2)(s-3) & (s-2)(s-3)(s-4) \end{pmatrix}. \quad (37)$$

Matrix		Elements of the first row			
		11	12	13	14
$M_2 \equiv (p^2{}_{\alpha\beta})$	..	4	0	$4(s-2)$	0
$M_3 \equiv (p^3{}_{\alpha\beta})$	..	2	2	$2(s-1)$	$2(s-3)$
$M_4 \equiv (p^4{}_{\alpha\beta})$	..	0	0	12	$4(s-4)$

(38)

Case B.—k *not a prime number or its power*

In this case the theory of confounding is not available to us to construct the $k \times k \times k$ Latin cubes of the second order. Hence we must resort to the general method of construction of Latin cubes of the second order described in Section 3. It will, however, be seen that there are too many arbitrary elements in this method and so, due to the lack of any symmetry, these Latin cubes, when used to generate the blocks of the fourth replication of the (k^3, k^2) lattices, do not in general lead to p.b.i.b. designs.

Nevertheless it has been found that, even for this non-prime case, Latin cubes of the second order of side k, of a *particular type* and constructed by the general method, do lead to cubic lattice designs which are amenable to solution by the general method presented in the next section. In other words, just as for the prime-power case when $k = s > 4$, the designs thus obtained satisfy all the relationships between the parameters of the first kind, and with regard to the parameters $p^\gamma{}_{\alpha\beta}$ of the second kind, they satisfy all relationships up to the matrix M_1 and the first rows of the matrices M_2, M_3, M_4.

In order to be able to construct suitable Latin cubes of the second order of side k of the type described above, we must start with a *cyclic* Latin square for the groups $g_1, g_2, \ldots, g_k$ of k numbers each as described at the end of Section 3. In addition, we must ensure that, as in Theorem 1 which holds for the prime case, the row and column contents in all layers are *kept* constant. The (k^3, k^2) lattices thus generated have four associate classes and satisfy all the necessary relationships.

The association scheme is governed by the same rules (*a*), (*b*), (*c*), (*d*) as given in Case A above, and the parameters of these designs are given exactly by the equations (36), (37) and (38) with k substituted for s.

As an illustration, we give below a $6 \times 6 \times 6$ Latin cube of the second order generated in the manner described above.

L_1

6	12	18	24	30	36
5	11	17	23	29	35
4	10	16	22	28	34
3	9	15	21	27	33
2	8	14	20	26	32
1	7	13	19	25	31

L_2

7	13	19	25	31	1
12	18	24	30	36	6
11	17	23	29	35	5
10	16	22	28	34	4
9	15	21	27	33	3
8	14	20	26	32	2

L_3

28	34	4	10	16	22
27	33	3	9	15	21
26	32	2	8	14	20
25	31	1	7	13	19
30	36	6	12	18	24
29	35	5	11	17	23

L_4

21	27	33	3	9	15
20	26	32	2	8	14
19	25	31	1	7	13
24	30	36	6	12	18
23	29	35	5	11	17
22	28	34	4	10	16

L_5

14	20	26	32	2	8
13	19	25	31	1	7
18	24	30	36	6	12
17	23	29	35	5	11
16	22	28	34	4	10
15	21	27	33	3	9

L_6

35	5	11	17	23	29
34	4	10	16	22	28
33	3	9	15	21	27
32	2	8	14	20	26
31	1	7	13	19	25
36	6	12	18	24	30

. (39)

It may readily be verified that the design obtained from (39) satisfies all the relevant combinatorial properties.

Lastly, it may be remarked that for both the prime and non-prime cases, the Latin cubes of the second order provide for the (k^3, k^2) lattices not only a convenient key to the blocks of the design, but also a *tactical configuration* for giving the association scheme of the varieties, which as stated by Rao (1947) must be annexed to the p.b.i.b. designs in general with equal λ's.

7. General Method of Solution for the (k^3, k^2) Lattices with Four Replicates

We give below the general solution for the (k^3, k^2) 3-dimensional lattice designs with four replicates. Whether k is a prime or non-prime, it is assumed that the fourth replication is generated in a suitable manner as indicated in Section 6, so that the p.b.i.b. solution for designs with four associate classes and any value of k applies. The method followed is similar to the one for designs with two and three associate classes as given by Bose and Nair (1939) and Rao (1947).

A. The P.B.I.B. Solution.

Consider a p.b.i.b. design with four associate classes and parameters:

$$\begin{array}{llll} v & b & r & k, \\ n_1 & n_2 & n_3 & n_4, \\ \lambda_1 = 1 & \lambda_2 = 0 & \lambda_3 = 0 & \lambda_4 = 0. \\ \end{array}$$

$$M_\gamma \equiv (p^\gamma{}_{\alpha\beta}), \quad \alpha, \ \beta, \ \gamma = 1, \ 2, \ 3, \ 4.$$

Let,

Q_i = Sum of the yields for the i-th variety *minus* the sum of the means of blocks in which it occurs,

Q_i' = Sum of the means of blocks in which the i-th variety occurs *minus* r times the grand mean m,

ΣQ_{ij} = Sum of the Q's for the j-th associates of the i-th variety,

v_i = Estimate of the i-th variety effect, and

S_{ij} = Sum of the varietal effects for the j-th associates of the i-th variety.

The Q-equations are then given by:

$$kQ_i = r(k-1)\,v_i - \lambda_1 S_{i1} - \lambda_2 S_{i2} - \lambda_3 S_{i3} - \lambda_4 S_{i4}.$$

Hence, on eliminating S_{i4} by using the constraining relation $\Sigma v_i = v_i + S_{i1} + S_{i2} + S_{i3} + S_{i4} = 0$, the normal equations for the intrablock solution are as follows:

$$\left.\begin{aligned} kQ_i &= A_{14}v_i + B_{14}S_{i1} + C_{14}S_{i2} + D_{14}S_{i3} \\ k\,\Sigma\, Q_{i1} &= A_{24}v_i + B_{24}S_{i1} + C_{24}S_{i2} + D_{24}S_{i3} \\ k\,\Sigma\, Q_{i2} &= A_{34}v_i + B_{34}S_{i1} + C_{34}S_{i2} + D_{34}S_{i3} \\ k\,\Sigma\, Q_{i3} &= A_{44}v_i + B_{44}S_{i1} + C_{44}S_{i2} + D_{44}S_{i3} \end{aligned}\right\} \qquad (40)$$

where,

$$\left.\begin{aligned}
A_{14} &= r(k-1)+\lambda_4,\ B_{14}=\lambda_4-\lambda_1,\ C_{14}=\lambda_4-\lambda_2=0,\\
&\qquad D_{14}=\lambda_4-\lambda_3=0.\\
A_{24} &= (\lambda_4-\lambda_1)(n_1-p_{11}{}^4)\\
B_{24} &= r(k-1)+\lambda_4+(\lambda_4-\lambda_1)(p_{11}{}^1-p_{11}{}^4)\\
C_{24} &= (\lambda_4-\lambda_1)(p^2{}_{11}-p_{11}{}^4)\\
D_{24} &= (\lambda_4-\lambda_1)(p_{11}{}^3-p_{11}{}^4)\\
A_{34} &= -(\lambda_4-\lambda_1)\,p_{12}{}^4\\
B_{34} &= (\lambda_4-\lambda_1)(p_{12}{}^1-p_{12}{}^4)\\
C_{34} &= r(k-1)+\lambda_4+(\lambda_4-\lambda_1)(p_{12}{}^2-p_{12}{}^4)\\
D_{34} &= (\lambda_4-\lambda_1)(p_{12}{}^3-p_{12}{}^4)\\
A_{44} &= -(\lambda_4-\lambda_1)\,p_{13}{}^4\\
B_{44} &= (\lambda_4-\lambda_1)(p_{13}{}^1-p_{13}{}^4)\\
C_{44} &= (\lambda_4-\lambda_1)(p_{13}{}^2-p_{13}{}^4)\\
D_{44} &= r(k-1)+\lambda_4+(\lambda_4-\lambda_1)(p_{13}{}^3-p_{13}{}^4)
\end{aligned}\right\}. \tag{41}$$

It may be noted that the normal equations (40) involve only the *first* rows of the matrices M_γ ($\gamma = 1, 2, 3, 4$). Solving these equations, the varietal effects are then estimated from:

$$v_i = \begin{vmatrix} kQ_i & B_{14} & C_{14} & D_{14} \\ k\Sigma Q_{i1} & B_{24} & C_{24} & D_{24} \\ k\Sigma Q_{i2} & B_{34} & C_{34} & D_{34} \\ k\Sigma Q_{i3} & B_{44} & C_{44} & D_{44} \end{vmatrix} \Big/ \Delta \tag{42}$$

where

$$\Delta = \begin{vmatrix} A_{14} & B_{14} & C_{14} & D_{14} \\ A_{24} & B_{24} & C_{24} & D_{24} \\ A_{34} & B_{34} & C_{34} & D_{34} \\ A_{44} & B_{44} & C_{44} & D_{44} \end{vmatrix}. \tag{43}$$

Let the cofactors of A_{14}, A_{24}, A_{34} and A_{44} in Δ be denoted by Δ_1, Δ_2, Δ_3, Δ_4. Then

$$v_i = \frac{(kQ_i)\,\Delta_1 + (k\Sigma Q_{i1})\,\Delta_2 + (k\Sigma Q_{i2})\,\Delta_3 + (k\Sigma Q_{i3})\,\Delta_4}{\Delta}$$

and the variances for the four associate classes and the mean variance are given by:

$$\left.\begin{aligned} V_1 &= \frac{2k\sigma^2}{\Delta}(\Delta_1 - \Delta_2) \\ V_2 &= \frac{2k\sigma^2}{\Delta}(\Delta_1 - \Delta_3) \\ V_3 &= \frac{2k\sigma^2}{\Delta}(\Delta_1 - \Delta_4) \\ V_4 &= \frac{2k\sigma^2}{\Delta}(\Delta_1) \end{aligned}\right\} \tag{44}$$

and

$$V_m = \frac{2k\sigma^2}{(v-1)\,\Delta}\left[(v-1)\,\Delta_1 - n_1\Delta_2 - n_2\Delta_3 - n_3\Delta_4\right] \tag{45}$$

where σ^2 is the intra-block error variance.

The adjusted sum of squares due to varieties is, as usual, given by $\Sigma\, v_i Q_i$, and the overall efficiency factor is given by:

$$E.F. = \frac{2\sigma^2}{r} \Big/ V_m = \frac{(v-1)\,\Delta}{rk\,\left[(v-1)\,\Delta_1 - n_1\Delta_2 - n_2\Delta_3 - n_3\Delta_4\right]}. \tag{46}$$

As indicated by Rao (1947), the combined intra- and inter-block solution is then obtained by making the substitutions R, Λ_i and P_i for r, λ_i and Q_i in the formulæ for estimates of varietal effects and variances derived above. The relationships to be used are:

$$\left.\begin{aligned} R &= r\left\{w + \frac{w'}{(k-1)}\right\} \\ \Lambda_i &= \lambda_i (w - w') \\ P_i &= wQ_i + w'Q_i' \end{aligned}\right\}. \tag{47}$$

B. Solution for the (k³, k²) *lattices in four replicates.*

We shall now derive the combined intra- and inter-block solution for the cubic lattice designs in blocks of k plots and four replicates by making use of the transformations (47). The parameters (*see* Section 6) of these designs are as follows:

$$\left.\begin{array}{l} v = k^3, \quad b = 4k^2, \quad r = 4, \quad k = k \\ n_1 = 4(k-1), \quad n_2 = 3(k-1), \\ n_3 = 6(k-1)(k-2), \quad n_4 = (k-1)(k-2)(k-3) \\ \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0 \end{array}\right\} . \tag{48}$$

Matrix		Elements of the first row			
		11	12	13	14
$p^1_{\alpha\beta}$	..	$(k-2)$	3	$3(k-2)$	0
$p^2_{\alpha\beta}$	..	4	0	$4(k-2)$	0
$p^3_{\alpha\beta}$	..	2	2	$2(k-1)$	$2(k-3)$
$p^4_{\alpha\beta}$	..	0	0	12	$4(k-4)$

(49)

Now, for brevity's sake, let us put

$$\left.\begin{array}{r} (k-1)w + w' = a \\ w - w' = b \end{array}\right\}$$

so that $(a + b) = kw$. Using (48) the transformations (47) then become:

$$R(k-1) = 4\{(k-1)w + w'\} = 4a$$

$$\Lambda_i = \lambda_i (w - w') = \lambda_i b$$

so that,

$$\Lambda_1 = (w - w') = b, \quad \Lambda_2 = 0, \quad \Lambda_3 = 0, \quad \Lambda_4 = 0 \tag{50}$$

Also, on substituting R and Λ_i for r and λ_i in (41), let the transformed values of A_{ij}, etc., be denoted by A'_{ij}, etc. Using (48), (49) and (50), the new coefficients of the normal equations are then as follows:

$$\left.\begin{aligned}
A'_{14} &= R(k-1) + A_4 = 4a,\ B'_{14} = A_4 - A_1 = -b,\\
&\qquad C'_{14} = 0,\ D'_{14} = 0,\\
A'_{24} &= (A_4 - A_1)[4(k-1) - 0] = -4(k-1).b\\
B'_{24} &= R(k-1) + A_4 + (A_4 - A_1)[(k-2) - 0]\\
&= 4a - (k-2).b\\
C'_{24} &= (A_4 - A_1)[4 - 0] = -4b\\
D'_{24} &= (A_4 - A_1)[2 - 0] = -2b\\
A'_{34} &= -(A_4 - A_1)[0] = 0\\
B'_{34} &= (A_4 - A_1)[3 - 0] = -3b\\
C'_{34} &= R(k-1) + A_4 + (A_4 - A_1)[0 - 0] = 4a\\
D'_{34} &= (A_4 - A_1)[2 - 0] = -2b\\
A'_{44} &= -(A_4 - A_1)[12] = 12b\\
B'_{44} &= (A_4 - A_1)[3(k-2) - 12] = -(3k-18).b\\
C'_{44} &= (A_4 - A_1)[4(k-1) - 12] = -(4k-20).b\\
D'_{44} &= R(k-1) + A_4 + (A_4 - A_1)[2(k-1) - 12]\\
&= 4a - (2k-14).b
\end{aligned}\right\} . \qquad (51)$$

Hence

$$\Delta' = \begin{vmatrix} A'_{14} & B'_{14} & C'_{14} & D'_{14} \\ A'_{24} & B'_{24} & C'_{24} & D'_{24} \\ A'_{34} & B'_{34} & C'_{34} & D'_{34} \\ A'_{44} & B'_{44} & C'_{44} & D'_{44} \end{vmatrix}$$

$$= \begin{vmatrix} 4a & -b & 0 & 0 \\ -4(k-1)\,b & 4a - (k-2)\,b & -4b & -2b \\ 0 & -3b & 4a & -2b \\ 12b & -(3k-18)\,b & -(4k-20)\,b & 4a - (2k-14)\,b \end{vmatrix} . \qquad (52)$$

Using the relation $(a + b) = kw$, we derive from (52) the following cofactors of the elements in the first column:

$$\left.\begin{aligned}
\Delta'_1 &= 8kw.[8a^2 - 6(k-4)\,ab + (k^2 - 10k + 22)\,b^2]\\
\Delta'_2 &= 8kw.b\,[2a - (k-5)\,b]\\
\Delta'_3 &= 8kw.2b^2\\
\Delta'_4 &= 8kw.b^2
\end{aligned}\right\} . \qquad (53)$$

Hence

$$\left.\begin{aligned} \Delta' &= 32kw\,[8a^3 - (6k-24)\,a^2b + (k^2 - 12k + 24)\,ab^2 \\ &\qquad + (k^2 - 6k + 8)\,b^3] \\ &= 32kw\,[\{(a+b)\}\,\{2a - (k-2)\,b\}\,\{4a - (k-4)\,b\}] \\ &= 32kw\,[kw.k\,(w+w').k\,(3w+w')] \\ &= 4k^4w.[(2w+2w')\,(3w+w')\,(4w)] \end{aligned}\right\}. \qquad (54)$$

Replacing Q_i by P_i and other symbols by primed ones in (42) and (43) we obtain the solution:

$$v'_i = \frac{(kP_i)\,\Delta'_1 + (k\,\Sigma\,P_{i1})\,\Delta'_2 + (k\,\Sigma\,P_{i2})\,\Delta'_3 + (k\,\Sigma\,P_{i3})\,\Delta'_4}{\Delta'}. \qquad (55)$$

Substituting for a and b in terms of w and w' in (53), we then obtain:

$$\left.\begin{aligned} \Delta'_1-\Delta'_2 &= 8kw.[3\,(k^2+k+1)\,w^2 + (4k^2-6)\,ww' + (k^2-3k+3)\,w'^2] \\ &= 8kw.F_1 \\ \Delta'_1-\Delta'_3 &= 8kw.[3k^2+4k+4)\,w^2 + (4k^2-8)\,ww' + (k^2-4k+4)\,w'^2] \\ &= 8kw.F_2 \\ \Delta_1'-\Delta'_4 &= 8kw.[(3k^2+4k+5)\,w^2 + (4k^2-10)\,ww' + (k^2-4k+5)\,w'^2] \\ &= 8kw.F_3 \\ \Delta'_1 &= 8kw.\;[(3k^2+4k+6)\,w^2 + (4k^2-12)\,ww' + (k^2-4k+6)\,w'^2] \\ &= 8kw.F_4, \text{ say} \end{aligned}\right\}. \qquad (56)$$

The variances V_j for the differences $(v_i' - v_j')$ between two varieties which are j-th associates $(j = 1, 2, 3, 4)$ are then given by:

$$V_j = \frac{4}{k^2} \cdot \frac{F_j}{[(2w+2w')\,(3w+w')\,(4w)]} \qquad (57)$$

By the method of partial fractions, we easily obtain from (56) and (57) the results:

$$\left.\begin{aligned} V_1 &= \frac{2}{k^2}\left[\frac{3}{(2w+2w')} + \frac{3\,(k-2)}{(3w+w')} + \frac{(k^2-3k+3)}{4w}\right] \\ V_2 &= \frac{2}{k^2}\left[\frac{4}{(2w+2w')} + \frac{4\,(k-2)}{(3w+w')} + \frac{(k^2-4k+4)}{4w}\right] \\ V_3 &= \frac{2}{k^2}\left[\frac{5}{(2w+2w')} + \frac{2\,(2k-5)}{(3w+w')} + \frac{(k^2-4k+5)}{4w}\right] \\ V_4 &= \frac{2}{k^2}\left[\frac{6}{(2w+2w')} + \frac{4\,(k-3)}{(3w+w')} + \frac{(k^2-4k+6)}{4w}\right] \end{aligned}\right\}. \qquad (58)$$

Hence,

$$V_m = \frac{2}{(k^2 + k + 1)} \left[\frac{6}{(2w + 2w')} + \frac{4(k-2)}{(3w + w')} + \frac{(k^2 - 3k + 3)}{4w} \right] \tag{59}$$

and the overall efficiency factor (with $w' = 0$) reduces to

$$E.F. \, (r = 4 \text{ replicates}) = \frac{3(k^2 + k + 1)}{3k^2 + 7k + 13}. \tag{60}$$

It is well known that the efficiency factor of the cubic lattice with 3 replicates and $w' = 0$ is given by (Yates, 1939):

$$E.F. \, (r = 3 \text{ replicates}) = \frac{2(k^2 + k + 1)}{2k^2 + 5k + 11}. \tag{61}$$

It is readily seen from (60) and (61) that

$$E.F. \, (r = 4) > E.F. \, (r = 3) \tag{62}$$

Hence, just as in the 2-dimensional lattices the efficiency is increased on using a Latin square for obtaining additional replications, similarly the use of a Latin cube of the second order increases the efficiency of the cubic lattice designs.

It may also be noted that the expression (59) for the mean variance is in accord with the general theory for prime-power designs ($k = s$) as presented by Kempthorne (1952). This follows easily from the fact that out of the $(k^2 + k + 1)$ sets of $(k - 1)$ d.f. for the comparisons between the k^3 varieties, 6 are confounded twice, $4(k - 2)$ are confounded once only, and the remaining $(k^2 - 3k + 3)$ sets are unconfounded.

Lastly it may be remarked that with $k = 3$, there are only 3 associate classes in the design. Hence the solution may be carried out with the formulæ for 3 associate classes given by Rao (1947). The parameters of the design are seen to be derivable from (48), (49) by putting $k = 3$. For details of calculation see the numerical example in Section 8. On carrying through these steps, the expressions for the variances V_1, V_2, V_3, V_m and the $E.F.$ are then found to be the same as those given in (58), (59), and (60) on making the substitution $k = 3$.

C. *Analysis of variance for the* (k^3, k^2) *designs in four replicates.*

The analysis of variance table is easily constructed as follows:

TABLE III

Analysis of variance

Source	D.F.	S.S.	M.S.	E (M.S.)
Replications ..	3	R	..	..
Varieties (ignoring blocks)	$k^3 - 1$	V	..	..
Blocks within replications (eliminating varieties)	$4(k^2-1)$	B	..	$\sigma^2+\frac{3}{4}.k\sigma_\beta^2$
Intra-block error ..	$3k^3-4k^2+1$	..	..	σ^2
TOTAL ..	$4k^3-1$	T	..	..

Here

$\sigma^2 = E$ (Intra-block error variance).

$\sigma_\beta^2 = E$ (Additional variance due to the variation among the incomplete block means freed of varietal effects).

The sums of squares T, R and V are obtained as usual. The sum of squares B for blocks (eliminating varieties) is obtained as follows:

We have,

S.S. blocks (ignoring varieties) + S.S. varieties (eliminating blocks)

= S.S. blocks (eliminating varieties) + S.S. varieties (ignoring blocks).

Hence,

$$B = S.S. \text{ blocks within replications (ignoring varieties)} + \Sigma\, v_i Q_i - V. \qquad (63)$$

The error sum of squares and d.f. are obtained by subtraction. The quantities σ^2 and σ_β^2 are estimated by equating the mean squares in the last two lines of Table III to their respective expectations, and finally w and w' are estimated from the relations:

$$\frac{1}{w} = \sigma^2, \quad \frac{1}{w'} = \sigma^2 + k\sigma_\beta^2. \qquad (64)$$

8. Numerical Example: 3^3 Lattice with Four Replications

As an illustration of the results discussed in the earlier sections we shall rework out the example given by Federer (1949) by the methods presented in this paper. He has used uniformity trial data with corn yield tests for 2×10 hill plots. Varieties are designated as usual by the symbols $(x_1\, x_2\, x_3)$ where $x_1,\ x_2,\ x_3 = 0,\ 1,\ 2$ and the fourth replication confounds the following effects:

Replication IV: $AB^2,\ AC,\ BC,\ ABC^2$

as given in (16). The corresponding blocks are then given in (18) and the 3×3×3 Latin cube of the second order thus generated in (20). The field arrangement of the varieties is given in Table IV.

To carry out the analysis of variance for the design, we obtain by the usual procedure the following values of sums of squares:

Total S.S. = 761·83.

Replicates S.S. = 181·41.

S.S. blocks (ignoring varieties) = 604·84.

Hence,

S.S. blocks within replications (ignoring varieties)

= 604·84 − 181·41 = 423·43.

S.S. varieties (ignoring blocks) = V = 115·22.

In order to calculate the S.S. for blocks within replications (eliminating varieties) we now need the value of $\Sigma v_i Q_i$. Using Rao's (1947) notation for p.b.i.b. designs with 3 associate classes the necessary calculations are presented in Table V.

The parameters for the design are:

$$v = 27,\ b = 36,\ r = 4,\ k = 3$$

$$n_1 = 8,\ n_2 = 6,\ n_3 = 12$$

$$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0$$

$$\Lambda_1 = (w - w'),\quad \Lambda_2 = 0,\quad \Lambda_3 = 0$$

$$R = 4\left[w + \frac{w'}{2}\right] = 4w + 2w'.$$

TABLE IV

Field arrangement showing plot yields in pounds of ear corn for 3^3 lattice
(Variety numbers in parenthesis)

Replicate I			Block Totals	Replicate II			Block Totals	Replicate III			Block Totals	Replicate IV			Block Totals
(112) 30·6	(111) 32·0	(110) 30·3	92·9	(001) 33·0	(011) 34·4	(021) 33·1	100·5	(100) 30·2	(200) 31·3	(000) 30·6	92·1	(022) 31·3	(210) 29·9	(101) 30·8	92·0
(002) 29·9	(000) 31·6	(001) 32·5	94·0	(201) 31·0	(221) 29·2	(211) 29·7	89·9	(210) 26·5	(010) 30·4	(110) 31·8	88·7	(122) 32·2	(201) 30·6	(010) 29·1	91·9
(210) 32·5	(211) 30·6	(212) 29·5	92·6	(120) 29·9	(110) 30·7	(100) 30·2	90·8	(002) 24·2	(102) 29·4	(202) 29·3	82·9	(020) 28·1	(102) 28·7	(211) 30·5	87·3
(120) 31·0	(122) 27·9	(121) 30·0	88·9	(102) 29·2	(112) 27·7	(122) 27·3	84·2	(020) 26·7	(120) 25·2	(220) 28·6	80·5	(121) 27·5	(200) 24·0	(012) 27·4	78·9
(100) 32·6	(102) 30·6	(101) 32·9	96·1	(202) 27·4	(212) 28·8	(222) 26·8	83·0	(201) 25·9	(101) 26·8	(001) 24·4	77·1	(112) 23·7	(221) 22·5	(000) 23·0	69·2
(011) 29·7	(010) 34·0	(012) 32·7	96·4	(020) 31·0	(000) 28·5	(010) 28·8	88·3	(121) 28·1	(221) 28·7	(021) 28·8	85·6	(120) 28·9	(011) 26·3	(202) 25·8	81·0
(220) 31·1	(221) 34·0	(222) 33·1	98·2	(012) 31·5	(002) 33·9	(022) 30·6	96·0	(212) 31·5	(112) 29·9	(012) 29·5	90·9	(001) 26·3	(110) 28·6	(222) 27·0	81·9
(020) 32·3	(021) 32·6	(022) 35·1	100·0	(101) 35·4	(121) 33·7	(111) 31·7	100·8	(122) 32·5	(022) 33·0	(222) 33·1	98·6	(111) 32·6	(002) 29·7	(220) 30·3	92·6
(202) 33·8	(201) 31·4	(200) 31·2	96·4	(200) 33·1	(210) 29·3	(220) 31·2	93·6	(211) 31·6	(011) 29·3	(111) 30·8	91·7	(100) 29·9	(021) 31·8	(212) 28·1	89·8
Replicate total ..			855·5	..			827·1	..			788·1	..			764·6
Grand total ..			..	..			..	..			..	..			3235·3

TABLE

Computations for the solution of the

Variety numbers (i)	Unadjusted totals T_i	$\Sigma B_{(i)}$	$kQ_i=3T_i-\Sigma B_{(i)}$	$kQ_i'=\Sigma B_{(i)}-T/9$	$k\Sigma Q_{i1}$	$k\Sigma Q_{i2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
000	113·7	343·6	− 2·5	−15·87778	−21·0	+17·3
001	116·2	353·5	− 4·9	− 5·97778	− 1·1	+14·1
002	117·7	365·5	−12·4	+ 6·02222	+ 7·9	− 6·0
010	122·3	365·3	+ 1·6	+ 5·82222	−18·6	+ 4·3
011	119·7	369·6	−10·5	+10·12222	+19·2	− 1·8
012	121·1	362·2	+ 1·1	+ 2·72222	−20·5	− 6·8
020	118·1	356·1	− 1·8	− 3·37778	+16·9	−38·9
021	126·3	375·9	+ 3·0	+16·42222	−12·5	+18·1
022	130·0	386·6	+ 3·4	+27·12222	−16·3	− 0·3
100	122·9	368·8	− 0·1	+ 9·32222	+24·3	−11·2
101	125·9	366·0	+11·7	+ 6·52222	− 2·3	+ 0·1
102	117·9	350·5	+ 3·2	− 8·97778	+ 3·3	+ 6·8
110	121·4	354·3	+ 9·9	− 5·17778	−11·8	− 9·1
111	127·1	378·0	+ 3·3	+18·52222	+ 5·3	− 8·8
112	111·9	337·2	− 1·5	−22·27778	+ 8·8	− 3·2
120	115·0	341·2	+ 3·8	−18·27778	+ 1·6	+ 7·2
121	119·3	354·2	+ 3·7	− 5·27778	+17·1	− 8·6
122	119·9	363·6	− 3·9	+ 4·12222	+13·9	+26·8
200	119·6	361·0	− 2·2	+ 1·52222	− 4·4	+ 2·4
201	118·9	355·3	+ 1·4	− 4·17778	+13·9	−20·1
202	116·3	343·3	+ 5·6	−16·17778	− 21·0	− 3·4
210	118·2	366·9	−12·3	+ 7·42222	+26·2	+ 2·2
211	122·4	361·5	+ 5·7	+ 2·02222	−19·0	+19·1
212	117·9	356·3	− 2·6	− 3·17778	− 0·2	+ 4·1
220	121·2	364·9	− 1·3	+ 5·42222	−23·0	+25·8
221	114·4	342·9	+ 0·3	−16·57778	+ 6·8	−12·1
222	120·0	361·7	− 1·7	+ 2·22222	+ 6·5	−18·0
Totals ..	3235·3 (=T)	9705·9 (=$3T$)	0	− 0·0000	0	0

V

3^3 *lattice in four replicates*

v_i	kP_i	$k\Sigma P_{i1}$	$k\Sigma P_{i2}$	v_i'	Adjusted means $m+v_i'$	Federer's values for adjusted means
(8)	(9)	(10)	(11)	(12)	(13)	(14)
−0·6102	−2·121079	− 9·731671	+ 6·908104	−0·909348	29·0471	29·0471
−0·6389	−2·272417	+ 0·795788	+ 7·351046	−0·672996	29·2835	29·2835
−1·5806	−4·165056	+ 3·449291	− 2·628638	−1·246456	28·7100	28·7100
−0·0565	+1·032162	− 7·241327	+ 0·411399	+0·136345	30·0928	30·0928
−1·1352	−3·150315	+ 7·528846	− 1·807158	−0·809874	29·1466	29·1466
−0·1880	+0·613611	− 6·141969	− 1·449607	+0·024721	29·9812	29·9812
−0·1037	−0·923367	+ 7·317279	−12·950654	−0·151327	29·8051	29·8051
+0·2704	+2·348045	− 3·740816	+ 5·189304	+0·669728	30·6262	30·6262
+0·2093	+3·299124	− 2·914013	− 1·023804	+0·969618	30·9261	30·9261
+0·3269	+0·661639	+ 7·021104	− 7·593313	+0·368284	30·3248	30·3247
+1·5537	+4·844878	+ 1·990828	+ 1·906434	+1·606410	31·5629	31·5629
+0·5065	+0·518315	+ 0·192366	+ 2·841513	+0·182182	30·1387	30·1386
+1·1343	+3·297610	− 4·634715	− 2·853166	+0·915368	30·8718	30·8718
+0·5028	+2·617168	+ 1·776610	− 6·249982	+0·856581	30·8131	30·8130
−0·0778	−2·228574	+ 1·445359	+ 0·317994	−0·670573	29·2859	29·2859
+0·5630	+0·044493	+ 0·519046	+ 5·344430	+0·050175	30·0066	30·0066
+0·7398	+0·981848	+ 7·489199	− 1·287603	+0·508983	30·4655	30·4654
−0·2324	−1·142938	+ 3·389073	+ 7·573685	−0·242217	29·7143	29·7142
−0·3593	−0·704944	− 2·884261	− 0·580851	−0·304685	29·6518	29·6518
+0·3426	+0·208021	+ 3·771411	− 8·260417	+0·133577	30·0900	30·0900
+0·4259	+0·872067	− 8·922016	+ 0·056114	+0·038677	29·9951	29·9951
−1·2593	−4·022870	+12·758835	+ 2·710756	−0·929017	29·0275	29·0275
+0·5398	+2·273717	− 7·090376	+ 6·791080	+0·562654	30·5191	30·5191
−0·3435	−1·206214	+ 0·051319	+ 2·128676	−0·374513	29·5820	29·5820
−0·4518	−0·077498	− 8·752006	+ 8·603287	−0·224327	29·7321	29·7321
+0·1083	−1·131114	+ 0·918164	− 3·632712	−0·351041	29·6054	29·6054
−0·1861	−0·466316	+ 1·638620	− 7·815941	−0·136932	29·8195	29·8195
0	−0·0000	− 0·0000	− 0·0090	−0·0000	..	..

Matrix	Elements of the first row		
	11	12	13
p^1_{ij} ..	1	3	3
p^2_{ij} ..	4	0	4
p^3_{ij} ..	2	2	4

Hence, by substituting the values of the parameters in the relations given by Rao (1947, p. 553), we have

$$\begin{pmatrix} A_{13} & B_{13} & C_{13} \\ A_{23} & B_{23} & C_{23} \\ A_{33} & B_{33} & C_{33} \end{pmatrix} = \begin{pmatrix} 8 & -1 & 0 \\ -6 & 9 & -2 \\ 2 & -1 & 10 \end{pmatrix}$$

so that,

$$F = \text{cofactor of } A_{13} = 88$$
$$G = \text{cofactor of } A_{23} = 10$$
$$H = \text{cofactor of } A_{33} = 2$$
$$\Delta = A_{13} F + A_{23} G + A_{33} H = 648.$$

The unadjusted totals T_i, the sum of the block totals containing the i-th variety $= \Sigma B_{(i)}$, and the expressions $kQ_i = 3T_i - \Sigma B_{(i)}$, are given in columns 2, 3, 4 of Table V.

The rules for determining the association scheme of the design have already been stated in Section 6, whence we easily derive from a table of first and second associates of each variety the expressions $\Sigma(kQ_{i1})$ and $\Sigma(kQ_{i2})$ in columns 6 and 7 by summing the values in column 4 over the first and second associates of the i-th variety but excluding the i-th variety itself. The solution for v_i is then given by:

$$v_i = \frac{F(kQ_i) + G(\Sigma kQ_{i1}) + H(\Sigma kQ_{i2})}{\Delta}$$

$$= \frac{88 \text{ (col. 4)} + 10 \text{ (col. 6)} + 2 \text{ (col. 7)}}{648}$$

and the values obtained are tabulated in column 8. From columns and 8 we easily obtain

$$\Sigma(kQ_i)\, v_i = 100 \cdot 19305$$

or

$$\Sigma v_i Q_i = 33 \cdot 40.$$

Hence, using (63),

S.S. blocks within replications (eliminating varieties)

$$= 423 \cdot 43 + 33 \cdot 40 - 115 \cdot 22 = 341 \cdot 61$$

as against the value 341·63 given by Federer. The small difference is obviously due to rounding errors. The analysis of variance Table VI of Federer is thus completely derived, and to estimate w and w' we have the relations:

M.S. for blocks eliminating varieties (D.F. = 32)

$$= 10 \cdot 676 = \sigma^2 + \tfrac{3}{4} \,.\, (3\sigma_\beta^2).$$

M.S. for Intra-block error (D.F. = 46) = 2.686 = σ^2.

Hence,

$$w = 0 \cdot 3723008$$

$$w' = 0 \cdot 0749681.$$

To derive the combined solution, we then have

$$\begin{pmatrix} A_{13}' & B_{13}' & C_{13}' \\ A_{23}' & B_{23}' & C_{23}' \\ A_{33}' & B_{33}' & C_{33}' \end{pmatrix} = \begin{pmatrix} (8w+4w') & -(w-w') & 0 \\ -6(w-w') & (9w+3w') & -2(w-w') \\ 2(w-w') & -(w-w') & (10w+2w') \end{pmatrix}$$

so that

$$\begin{aligned} F' &= 88w^2 + 52ww' + 4w'^2 &&= 13 \cdot 6713304 \\ G' &= 10w^2 - 8ww' - 2w'^2 &&= 1 \cdot 1515530 \\ H' &= 2w^2 - 4ww' + 2w'^2 &&= 0 \cdot 1768135 \\ \Delta' &= 27 \,.\, (2w + 2w')(3w + w')(4w) &&= 42 \cdot 8692112. \end{aligned}$$

The values of $kQ_i' = \Sigma B_{(i)} - krm = \Sigma B_{(i)} - (T/9)$, where m is the general mean 29·956481 and T is the grand total 3235·3, are given in column 5. The values of $kP_i = w(kQ_i) + w'(kQ_i')$ are then tabulated in column 9 using the values of w and w' derived above on columns 4 and 5. The expressions $\Sigma(kP_{i1})$ and $\Sigma(kP_{i2})$ are then worked out as before from the kP_i values. The solution v_i' is then given by

$$v_i' = \frac{F'(kP_i) + G'\Sigma(kP_{i1}) + H'\Sigma(kP_{i2})}{\Delta'}$$

and is tabulated in column 12. The next column gives the adjusted means $m + v_i'$ for all the 27 varieties and may be compared with column 14 giving Federer's values for the same. It will be seen that but for a small difference in the fourth decimal place in a few cases, due to rounding errors, the columns 13 and 14 are identical.

Lastly, the variances for the three associate classes may be derived from the following steps. We have

$$F' - G' = 78w^2 + 60ww' + 6w'^2 = 12{\cdot}5197774$$
$$F' - H' = 86w^2 + 56ww' + 2w'^2 = 13{\cdot}4945169$$
$$F' = 88w^2 + 52ww' + 4w'^2 = 13{\cdot}6713304.$$

Hence,

$$V_1 = 2k \cdot \frac{(F' - G')}{\Delta'} = \frac{2}{9} \cdot \frac{(78w^2 + 60ww' + 6w'^2)}{(2w + 2w')(3w + w')(4w)}$$
$$= \frac{2}{9}\left[\frac{3}{2w + 2w'} + \frac{3}{3w + w'} + \frac{3}{4w}\right] = 1{\cdot}7523.$$

Similarly,

$$V_2 = 2k \cdot \frac{(F' - H')}{\Delta'} = \frac{2}{9}\left[\frac{4}{2w + 2w'} + \frac{4}{3w + w'} + \frac{1}{4w}\right]$$
$$= 1{\cdot}8887$$

$$V_3 = 2k \cdot \frac{F'}{\Delta'} = \frac{2}{9}\left[\frac{5}{2w + 2w'} + \frac{2}{3w + w'} + \frac{2}{4w}\right] = 1{\cdot}9134$$

so that,

$$SE_1 = 1{\cdot}324, \quad SE_2 = 1{\cdot}374, \quad SE_3 = 1{\cdot}383.$$

Finally,

$$V_m = \frac{8V_1 + 6V_2 + 12V_3}{26}$$
$$= \frac{2}{13}\left[\frac{6}{2w + 2w'} + \frac{4}{3w + w'} + \frac{3}{4w}\right] = 1{\cdot}8581$$

so that

$$\text{Average } SE = \sqrt{V_m} = 1{\cdot}363$$

and

$$E.F. = \frac{39}{61} = 63 \cdot 9\%.$$

We also have

Randomised block error variance $= 5 \cdot 96$

Average effective error variance $= (4/2)\, V_m = 2V_m = 3 \cdot 7162.$

Hence,

$$\text{Actual efficiency} = \frac{5 \cdot 96}{2V_m} = 160 \cdot 4\%$$

and

$$C.V. = \frac{\sqrt{2V_m}}{m} = 6 \cdot 4\%$$

so that all the results are seen to be in complete agreement with those given by Federer.

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